QUADRATIC INTEGRALS OF LINEAR MECHANICAL SYSTEMS

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1. We consider the system of linear differential equations with constant coefficients

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (s = 1, \ldots, n) \tag{1.1}$$

Suppose that the quadratic form

$$V = \sum_1^n a_{ij}x_ix_j$$

with constant coefficients has a derivative which, with the aid of (1.1), can be written as

$$\frac{dV}{dt} = \sum_1^n b_{ij}x_ix_j$$

The connection between the introduced coefficients has the following matrix form:

$$AP + (AP)_c = B \qquad (A = \| a_{ij} \|, \; B = \| b_{ij} \|, \; P = \| p_{ij} \|) \tag{1.2}$$

Here c indicates the adjoint (transposed matrix).

Let us take the system (1.1) in the particular form to which many systems of equations in mechanics can be reduced

$$\frac{dx_{2s-1}}{dt} = x_{2s}, \qquad \frac{dx_{2s}}{dt} = T_s x_1 + x_{2s+1} \qquad (s = 1, \ldots, k; \; x_{2k+1} \equiv 0) \tag{1.3}$$

The characteristic equation of the system (1.3) has the form

$$\varkappa^{2k} - T_1\varkappa^{2k-2} - \ldots - T_k = 0 \tag{1.4}$$

Let us set ourselves the problem of finding for the system (1.3) integrals (solutions) of the type

$$V = \sum_{1}^{2k} a_{ij} x_i x_j$$

From (1.2) one obtains the equation for the determination of A

$$AP + (AP)_c = 0 \qquad (1.5)$$

We have

$$AP = \left\| \begin{matrix} T_1 a_{12} + T_2 a_{14} + \ldots + T_k a_{12k} & a_{11} & a_{12} & \ldots & a_{12k-1} \\ T_1 a_{22} + T_2 a_{24} + \ldots + T_k a_{22k} & a_{21} & a_{22} & \ldots & a_{22k-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ T_1 a_{2k2} + T_2 a_{2k4} + \ldots + T_k a_{2k2k} & a_{2k1} & a_{2k2} & \ldots & a_{2k2k-1} \end{matrix} \right\| \qquad (1.6)$$

The determination of the matrix A is most easily accomplished by the direct use of the table (1.6). Taking into consideration (1.5), we obtain

$$AP = \left\| \begin{matrix} 0 & a_{11} & 0 & -a_{22} & 0 & a_{33} & \ldots \\ T_1 a_{22} - T_2 a_{33} + \ldots + (-1)^{k+1} T_k a_{k+1,\, k+1} & 0 & a_{22} & 0 & -a_{33} & \ldots & \\ 0 & -a_{22} & 0 & a_{33} & \ldots & & \\ -T_1 a_{33} + T_2 a_{44} - \ldots + (-1)^{k+2} T_k a_{k+2,\, k+2} & 0 & -a_{33} & \ldots & & & \\ 0 & a_{33} & \ldots & & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (-1)^{k+1} T_1 a_{k+1,\, k+1} + \ldots + (-1)^{2k} T_k a_{2k,\, 2k} & \ldots & & & & & \end{matrix} \right\| \qquad (1.7)$$

Introducing the notation $a_j = (-1)^j a_{jj}$, one can express the equations for the determination of these remaining unknowns in the form

$$a_j - T_1 a_{j+1} - T_2 a_{j+2} - \ldots - T_k a_{j+k} = 0 \qquad (j = 1, \ldots, k) \qquad (1.8)$$

By means of these equations the unknowns $a_1, \ldots, a_k$ can always be expressed in terms of the remaining $a_{k+1}, \ldots, a_{2k}$ which remain arbitrary. The matrix A will take on the following form:

$$A = \left\| \begin{matrix} -a_1 & 0 & -a_2 & 0 & \ldots & -a_k & 0 \\ 0 & a_2 & 0 & a_3 & \ldots & 0 & a_{k+1} \\ -a_2 & 0 & -a_3 & 0 & \ldots & -a_{k+1} & 0 \\ 0 & a_3 & 0 & a_4 & \ldots & 0 & a_{k+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_k & 0 & -a_{k+1} & 0 & \ldots & -a_{2k-1} & 0 \\ 0 & a_{k+1} & 0 & a_{k+2} & \ldots & 0 & a_{2k} \end{matrix} \right\| \qquad (1.9)$$

It breaks up into two Hankel matrices which are easily constructed from the rows and columns of A

$$A_1 = -\begin{Vmatrix} a_1 & a_2 & \dots & a_k \\ a_2 & a_3 & \dots & a_{k+1} \\ \dots & \dots & \dots & \dots \\ a_k & a_{k+1} & \dots & a_{2k+1} \end{Vmatrix}, \qquad A_2 = \begin{Vmatrix} a_2 & a_3 & \dots & a_{k+1} \\ a_3 & a_4 & \dots & a_{k+2} \\ \dots & \dots & \dots & \dots \\ a_{k+1} & a_{k+2} & \dots & a_{2k} \end{Vmatrix}$$

This corresponds to the breaking up of the integral V into two independent forms with even and odd subscripts on the variables

$$V = V_1 + V_2, \qquad V_1 = -\sum_1^{2k-1} a_{1/2\,(i+j)} x_i x_j \qquad (i, j \text{ — odd}) \tag{1.10}$$

$$V_2 = \sum_2^{2k} a_{1/2\,(i+j)} x_i x_j \qquad (i, j \text{ — even}) \tag{1.11}$$

Let us now return to the solution of Equations (1.8). For these equations one can obtain a fundamental system of solutions after one has assigned k systems of values to $a_{k+1}, a_{k+2}, \dots, a_{2k}$ arranged in a matrix of the following tyep, for example

$$A' \equiv \begin{Vmatrix} a_{k+1}^{(1)} & \dots & a_{2k}^{(1)} \\ a_{k+1}^{(2)} & \dots & a_{2k}^{(2)} \\ \dots & \dots & \dots \\ a_{k+1}^{(k)} & \dots & a_{2k}^{(k)} \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{Vmatrix}$$

In accordance with this we obtain k linearly independent quadratic integrals $V^{(1)}, \dots, V^{(k)}$; every other integral will be a linear combination of these integrals of the form

$$V = \lambda_1 V^{(1)} + \dots + \lambda_k V^{(k)} \qquad (\lambda_j = \text{const}) \tag{1.13}$$

The absence of nonlinear relations among the integrals $V^{(j)}$ follows from the fact that a certain Jacobean does not vanish, i.e. that, for example

$$\frac{\partial (V^{(1)} \dots V^{(k)})}{\partial (x_2, \dots x_{2k})} = |A'| \neq 0 \qquad \text{if} \quad x_2 = \dots = x_{2k-2} = 0, \qquad x_{2k} = \frac{1}{2}$$

2. It can be easily seen that Equation (1.8) can be satisfied by setting

$$a_j = \mu_m^{2k-j} \qquad (j = 1, \dots, 2k) \tag{2.1}$$

where μ_m is the square ($\mu_m = \kappa_m^2$) of an arbitrary root κ_m of the characteristic equation (1.4). In case of the absence of multiple roots κ_m, we will have a fundamental system of solutions of Equation (1.8) if we select the matrix A' as

$$A' = \left\| \begin{matrix} \mu_1^{k-1} & \mu_1^{k-2} & \ldots & \mu_1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu_k^{k-1} & \mu_k^{k-2} & \ldots & \mu_k & 1 \end{matrix} \right\| \tag{2.2}$$

Let us make the change of variables

$$\xi_s = x_{2k-2s+1}, \qquad \eta_s = x_{2k-2s+2} \qquad (s = 1, \ldots, k) \tag{2.3}$$

Then Formulas (1.10) and (1.11) take on the following form in terms of the new variables:

$$V_1 = -\sum_1^k a_{2k-i-j+1}\xi_i\xi_j, \qquad V_2 = \sum_1^k a_{2k-i-j+2}\eta_i\eta_j \tag{2.4}$$

In the case under consideration, when one selects the values (1.14) for the a_j, one obtains

$$V_1^{(m)} = -\sum_1^k \mu_m^{i+j-1}\xi_i\xi_j = -\mu_m(\xi_1 + \mu_m\xi_2 + \cdots + \mu_m^{k-1}\xi_k)^2$$
$$V_2^{(m)} = \sum_1^k \mu_m^{i+j-2}\eta_i\eta_j = (\eta_1 + \mu_m\eta_2 + \cdots + \mu_m^{k-1}\eta_k)^2 \tag{2.5}$$

One more linear substitution

$$u_m = \xi_1 + \mu_m\xi_2 + \cdots + \mu_m^{k-1}\xi_k, \qquad v_m = \eta_1 + \mu_m\eta_2 + \cdots + \mu_m^{k-1}\eta_k \qquad (m = 1, \ldots, k) \tag{2.6}$$

which is nonsingular for simple roots κ_m, leads to differential equations for the functions u_m, v_m which have the Hamiltonian structure

$$\frac{du_m}{dt} = \frac{\partial K}{\partial v_m} \qquad \frac{dv_m}{dt} = -\frac{\partial K}{\partial u_m} \tag{2.7}$$

where

$$2K = -\sum_1^k \mu_m u_m^2 + \sum_1^k v_m^2 = \sum V^{(m)} \tag{2.8}$$

We note that if the individual integrals $V^{(m)}$ for real variables ξ_i, η_i are complex numbers, then (for real T_j, which will always be assumed to be the case in the sequel) the integral K will take on only real values for arbitrary μ_m.

We shall give a more explicit form to this integral. Introducing the usual notation in the Newton sums

$$s_j = \sum_{m=1}^k \mu_m^j, \qquad s_0 = k \tag{2.9}$$

we obtain

$$2K = -\sum_1^k s_{i+j-1}\xi_i\xi_j + \sum_1^k s_{i+j-2}\eta_i\eta_j \tag{2.10}$$

The integral K is closely related to the problems on the stability of motion [1,2]. If the equations of the first approximation of the perturbed motion have the form of the system (1.3), then one can easily see that it is necessary and sufficient, in order to have stability of the first approximation, that the integral K be positive-definite (> 9). Indeed, since the elementary divisors of the system (1.3) are always relatively prime, stability implies the pure imaginary nature and simplicity of the roots of the characteristic equation (1.4), and, hence, in view of (2.8), the positiveness of K. The converse of this statement follows directly from the generally known theorem of Liapunov on the stability of motion.

As could be expected, Equations (1.8) for the determination of a_j in the construction of the integral K lead to some of the algebraic formulas of Newton

$$s_j = \sum_{r=1}^{k} T_r s_{j-r} \qquad \begin{pmatrix} j = k,\ k+1,\ \ldots,\ 2k-1 \\ s_0 = k \end{pmatrix}$$

which must be augmented with the remaining equations

$$s_j = \sum_{r=1}^{j} T_r s_{j-r} \qquad \begin{pmatrix} j = 1,\ 2,\ \ldots,\ k-1 \\ s_0 = j \end{pmatrix}$$

The conditions for the positive-definiteness of K are equivalent to known algebraic conditions [3,4] on the simplicity and pure imaginary nature of the roots of Equation (1.4).

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